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ESTIMATION OF THE RATE OF A DISCRETE-TIME MULTIVARIATE POINT PROCESS

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Abstract

We introduce the notion of a discrete-time, multivariate point process which can arise in the modeling of an optical communication system. We wish to estimate the rate of this process at time t given the past of the process up to time $t-1$. This requires the computation of a certain conditional expectation; we perform this computation by introducing an absolutely continuous change of measure and then applying the generalized Bayes' rule.

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I. Introduction

Suppose that a laser beam, whose intensity is modulated by an information source, strikes a photodetector which is a part of the receiver of an optical communication system. Suppose that the beam is also influenced by external random phenomena; this occurs, for example, if the beam passes through a turbulent atmosphere [1]. The occurrence of photoelectrons at the detector can then be modeled as a doubly-stochastic, time-space Poisson process [1]. It is often necessary to compute the conditional expectation of the rate of such a process, for example, in performing a likelihood ratio test (Snyder [2], Chapter 2 and Chapter 6). In general, the computation of this conditional expectation is quite difficult, although results are available if the rate process has a Gaussian form, and the photodetector surface is very large; in addition, results for *linear* estimates are available when the photodetector surface is arbitrary, but the rate process still has a Gaussian form [3]. (see also [4, 5] for related filtering results).

Let D represent the photosensitive surface of the photodetector. It is well known that the probability is negligible that more than one photoelectron will occur in the entire region D during a time interval of length Δt, if Δt is sufficiently small. Let $D_1, \ldots, D_K$, be a partition of D into disjoint subregions. Let Δt be "sufficiently small," and let $n_t(k)$ denote the number of photoelectrons occurring in the region D_k, during an interval $(t, t+\Delta t]$. We should expect that for each $1 \leq k \leq K$, $n_t(k)$ takes only the values 0 or 1. In addition, we can have $n_t(k) = 1$ for at most one k; for $i \neq k$, we must have $n_t(i) = 0$. In this paper we formulate the discrete-time version of the rate-estimation problem by introducing the notion of a discrete-time, multivariate point process. By adapting the continuous-time procedures found in (Bremaud [6], Chapter IV; see also pp. 69-70, 80 of Chapter III), we solve the problem of estimating the rate of a discrete-time, multivariate point process. (For more general discrete-time procedures, see [7, 8]).

II. Problem Statement

Let $K \geq 1$ be a fixed integer. Let $(\Omega, \mathbb{F}, \mathbf{P})$ be a probability space on which all the random variables in this section are defined. We call $\{ n_t(k);\ t \geq 1,\ 1 \leq k \leq K \}$ a discrete-time, multivariate point process if each $n_t(k)$ takes only the values 0 and 1 and if the events $\{n_t(k) = 1\}$, $1 \leq k \leq K$, are disjoint, so that simultaneous events do not occur. Next, let $\mathbb{G}_0$ denote the trivial σ-field on Ω, and set

$$\mathbb{G}_t \triangleq \sigma\{ n_s(k);\ 1 \leq s \leq t,\ 1 \leq k \leq K \}.$$

Now, let $\mathbb{X}$ be a sub-σ-field of $\mathbb{F}$, and let

$$\mathbb{F}_t \triangleq \mathbb{G}_t \vee \mathbb{X}\ ;\quad t \geq 0,$$

denote the smallest σ-field containing $\mathbb{G}_t \bigcup \mathbb{X}$. We assume that the random variables

$$\lambda_t(k) \triangleq \mathbf{E}\,[\, n_t(k) \mid \mathbb{F}_{t-1} \,]\ ;\quad t \geq 1,$$

are actually $\mathbb{X}$-measurable (note that $\mathbb{X} = \mathbb{F}_0$) with *known* joint distributions. For example, we might take $\mathbb{X} = \sigma(X)$ for some random variable X with known distribution function $F(x)$. Then each $\lambda_t(k)$ would be some Borel function of X. Clearly, the joint distributions of the $\{ \lambda_t(k);\ t \geq 1,\ 1 \leq k \leq K \}$ would be known, at least in principle. Now, observe that $\lambda_t(k) = \mathbf{P}(\, n_t(k) = 1 \mid \mathbb{F}_{t-1} \,)$; since we assume that simultaneous events do not occur,

$$\sum_{k=1}^{K} \lambda_t(k) = \mathbf{P}\left(\bigcup_{k=1}^{K} \{ n_t(k) = 1 \} \mid \mathbb{F}_{t-1} \right).$$

This implies that $p_t \triangleq 1 - \sum_{k=1}^{K} \lambda_t(k)$ is also a conditional probability, and hence, nonnegative and bounded above by 1.

Our objective is to compute

$$\hat{\lambda}_t(k) \triangleq \mathbf{E}\,[\, n_t(k) \mid \mathbb{G}_{t-1} \,]$$

in terms of the known joint distribution of $\{ \lambda_s(k);\ 1 \leq s \leq t,\ 1 \leq k \leq K \}$.

III. A Reformulation of the Problem

We shall solve our problem by reformulating the probabilistic setting above on a different probability space, $(\tilde{\Omega}, \tilde{\boldsymbol{F}}, \mathbf{Q})$. On $(\tilde{\Omega}, \tilde{\boldsymbol{F}}, \mathbf{Q})$ let $\{ \nu_t(k);\ t \geq 1,\ 1 \leq k \leq K \}$ be a discrete-time, multivariate point process. In a manner analogous to that outlined in the previous section, we take $\tilde{\boldsymbol{G}}_0$ to be the trivial σ-field on $\tilde{\Omega}$, and set

$$\tilde{\boldsymbol{G}}_t \triangleq \sigma\{ \nu_s(k);\ 1 \leq s \leq t,\ 1 \leq k \leq K \}\ ; \quad t \geq 1.$$

Next, let $\tilde{\boldsymbol{X}}$ be a sub-σ-field of $\tilde{\boldsymbol{F}}$, and set

$$\tilde{\boldsymbol{F}}_t \triangleq \tilde{\boldsymbol{G}}_t \vee \tilde{\boldsymbol{X}}\ ; \quad t \geq 0.$$

We now assume that

$$\mathbf{E}_{\mathbf{Q}} [\, \nu_t(k) \mid \tilde{\boldsymbol{F}}_{t-1} \,] = \mu_t(k), \tag{1}$$

where the $\{ \mu_t(k) \}$ are arbitrary constants satisfying $\mu_t(k) > 0$ and $q_t \triangleq 1 - \sum_{k=1}^{K} \mu_t(k) > 0$. (The symbol $\mathbf{E}_{\mathbf{Q}}$ denotes expectation with respect to the measure $\mathbf{Q}$). As a consequence of the above assumption, under the measure $\mathbf{Q}$, each $\nu_t(k)$ is independent of $\tilde{\boldsymbol{G}}_{t-1}$. In addition, the σ-field $\tilde{\boldsymbol{G}}_t$ is independent of the σ-field $\tilde{\boldsymbol{X}}$ (see Appendix). Next, let the random variables $\{ \tilde{\lambda}_t(k);\ t \geq 1,\ 1 \leq k \leq K \}$ defined on $\tilde{\Omega}$ be $\tilde{\boldsymbol{X}}$-measurable *and* have the *same* joint distributions under $\mathbf{Q}$ as $\{ \lambda_t(k);\ t \geq 1,\ 1 \leq k \leq K \}$ (defined on Ω) under $\mathbf{P}$.

Given the preceding probabilistic setting on $(\tilde{\Omega}, \tilde{\boldsymbol{F}}, \mathbf{Q})$, we make the following definitions. Let

$$\tilde{p}_t \triangleq 1 - \sum_{k=1}^{K} \tilde{\lambda}_t(k)\ ; \quad t \geq 1,$$

$$l_t \triangleq \sum_{k=1}^{K} \left(\frac{\tilde{\lambda}_t(k)}{\mu_t(k)} - \frac{\tilde{p}_t}{q_t} \right) \nu_t(k) + \frac{\tilde{p}_t}{q_t}\ ; \quad t \geq 1, \tag{2}$$

and

$$L_t \triangleq \prod_{s=1}^{t} l_s \ ; \quad t \geq 1,$$

with $L_0 \equiv 1$. Observe that the denominators in (2) are nonzero and deterministic. Also, $0 \leq \tilde{\lambda}_t(k) \leq 1$ Q-a.s., and hence, $\tilde{p}_t$ is clearly bounded and Q-integrable. (Recall that the joint distributions of the $\{ \tilde{\lambda}_t(k) \}$ under Q are the same as those of the $\{ \lambda_t(k) \}$ under P). Consequently, l_t and L_t are Q-integrable. Since $\sum_{k=1}^{K} \tilde{\lambda}_t(k) = 1 - \tilde{p}_t$, and $\sum_{k=1}^{K} \mu_t(k) = 1 - q_t$, it is easy to see that $\mathbf{E_Q} [\, l_t \mid \tilde{\boldsymbol{F}}_{t-1}] = 1$. Since $L_t = L_{t-1} l_t$, it is clear that

$$\mathbf{E_Q} [\, L_t \mid \tilde{\boldsymbol{F}}_{t-1}] = L_{t-1} \ ; \tag{3}$$

i.e., L_t is an $\tilde{\boldsymbol{F}}_t$-martingale under Q. Since, $\mathbf{E_Q} [\, L_t \,] = \mathbf{E_Q} [\, L_1 \,] = \mathbf{E_Q} [\, l_1 \,] = 1$, we can define a new measure $\tilde{\mathbf{P}}$ on $\tilde{\boldsymbol{F}}_t$ by

$$\tilde{\mathbf{P}}(F) \triangleq \int_F L_t \, d\mathbf{Q} \ ; \quad F \in \tilde{\boldsymbol{F}}_t \, .$$

(Technically, we should show that the family $\{ L_t \}$ is uniformly integrable. However, for our purposes this is not necessary since if we wish to compute $\mathbf{E} [\, n_\tau(k) \mid \boldsymbol{G}_{\tau-1}]$ for some τ, we can select any finite $T \geq \tau$ and then restrict our attention to $1 \leq t \leq T$). Observe that since $L_0 = 1$, $\tilde{\mathbf{P}} = \mathbf{Q}$ on $\tilde{\boldsymbol{X}}$. If $\tilde{\mathbf{E}}$ denotes expectation with respect to the measure $\tilde{\mathbf{P}}$, it is not hard to verify (since simultaneous events do not occur) that

$$\begin{aligned} \tilde{\mathbf{E}} [\, \nu_t(k) \mid \tilde{\boldsymbol{F}}_{t-1}] &= \mathbf{E_Q} [\, \nu_t(k) \frac{L_t}{L_{t-1}} \mid \tilde{\boldsymbol{F}}_{t-1}] \\ &= \mathbf{E_Q} [\, \nu_t(k) \, l_t \mid \tilde{\boldsymbol{F}}_{t-1}] \\ &= \tilde{\lambda}_t(k), \end{aligned} \tag{4}$$

where the first equality in (4) follows from the generalized Bayes' rule. Now, since $\tilde{\mathbf{P}} = \mathbf{Q}$ on $\tilde{\boldsymbol{X}}$,

$$\tilde{\lambda}_t(k) = \tilde{\mathbf{E}} [\, \nu_t(k) \mid \tilde{\boldsymbol{F}}_{t-1}]$$

under $\tilde{\mathbf{P}}$ is probabilistically equivalent to $\lambda_t(k)$ under $\mathbf{P}$. In fact, we can make the following

statement. The probabilistic relationships among

$$\{ n_t(k) \},\ \{ \lambda_t(k) \},\ \{ \mathcal{F}_t \},\ \{ \mathcal{G}_t \},\ \text{and}\ \mathcal{X}$$

under $\mathbf{P}$ are the same as those among

$$\{ \nu_t(k) \},\ \{ \tilde{\lambda}_t(k) \},\ \{ \tilde{\mathcal{F}}_t \},\ \{ \tilde{\mathcal{G}}_t \},\ \text{and}\ \tilde{\mathcal{X}}$$

under $\tilde{\mathbf{P}}$. In the next section we shall compute $\tilde{\mathrm{E}}\,[\, \nu_t(k) \mid \tilde{\mathcal{G}}_{t-1} \,]$ explicitly as a Borel function of $\{ \nu_s(i);\ 1 \le s \le t-1,\ 1 \le i \le K \}$. From the statement above, it follows that $\mathrm{E}\,[\, n_t(k) \mid \mathcal{G}_{t-1} \,]$ will be equal to the same Borel function applied to $\{ n_s(i);\ 1 \le s \le t-1,\ 1 \le i \le K \}$.

IV. Calculations

In this section we compute $\tilde{\mathrm{E}}\,[\, \nu_t(k) \mid \tilde{\mathcal{G}}_{t-1} \,]$. By Bayes' rule,

$$\tilde{\mathrm{E}}\,[\, \nu_t(k) \mid \tilde{\mathcal{G}}_{t-1} \,] = \frac{\mathrm{E_Q}\,[\, \nu_t(k)\, L_t \mid \tilde{\mathcal{G}}_{t-1} \,]}{\mathrm{E_Q}\,[\, L_t \mid \tilde{\mathcal{G}}_{t-1} \,]}\,. \tag{5}$$

Next, observe that

$$\mathrm{E_Q}\,[\, L_t \mid \tilde{\mathcal{G}}_{t-1} \,] = \mathrm{E_Q}\,[\, \mathrm{E_Q}\,[\, L_t \mid \tilde{\mathcal{F}}_{t-1} \,] \mid \tilde{\mathcal{G}}_{t-1} \,] = \mathrm{E_Q}\,[\, L_{t-1} \mid \tilde{\mathcal{G}}_{t-1} \,]$$

by equation (3). In addition,

$$\begin{aligned} \mathrm{E_Q}\,[\, \nu_t(k)\, L_t \mid \tilde{\mathcal{G}}_{t-1} \,] &= \mathrm{E_Q}\,[\, \mathrm{E_Q}\,[\, \nu_t(k)\, L_t \mid \tilde{\mathcal{G}}_t \,] \mid \tilde{\mathcal{G}}_{t-1} \,] \\ &= \mathrm{E_Q}\,[\, \nu_t(k)\, \mathrm{E_Q}\,[\, L_t \mid \tilde{\mathcal{G}}_t \,] \mid \tilde{\mathcal{G}}_{t-1} \,]. \end{aligned}$$

Hence, (5) can be rewritten as

$$\tilde{\mathrm{E}}\,[\, \nu_t(k) \mid \tilde{\mathcal{G}}_{t-1} \,] = \frac{\mathrm{E_Q}\,[\, \nu_t(k)\, \mathrm{E_Q}\,[\, L_t \mid \tilde{\mathcal{G}}_t \,] \mid \tilde{\mathcal{G}}_{t-1} \,]}{\mathrm{E_Q}\,[\, L_{t-1} \mid \tilde{\mathcal{G}}_{t-1} \,]}\,. \tag{6}$$

Before proceeding further, it will be convenient to introduce the following notation. For each $t \ge 1$, let

$$\nu_t \triangleq [\, \nu_t(1), \ldots, \nu_t(K) \,]'\,,$$

where $'$ denotes transpose. Then ν_t is a K-dimensional random vector. Now, let

$$\overline{\nu}_t \triangleq [\, \nu_1, \ldots, \nu_t \,]' .$$

Clearly, $\overline{\nu}_t$ is a tK-dimensional random vector. Next, let

$$z_t \triangleq [\, z_t(1), \ldots, z_t(K) \,]' \text{ and } \overline{z}_t \triangleq [\, z_1, \ldots, z_t \,]'$$

denote "dummy" variables in $\mathbb{R}^K$ and $\mathbb{R}^{tK}$, respectively. Define

$$h_t(z_t) = \sum_{k=1}^{K} \left\{ \frac{\lambda_t(k)}{\mu_t(k)} - \frac{p_t}{q_t} \right\} z_t(k) + \frac{p_t}{q_t}, \tag{7}$$

and

$$\tilde{h}_t(z_t) = \sum_{k=1}^{K} \left\{ \frac{\tilde{\lambda}_t(k)}{\mu_t(k)} - \frac{\tilde{p}_t}{q_t} \right\} z_t(k) + \frac{\tilde{p}_t}{q_t} .$$

Clearly, $h_t(z_t)$ is an $\boldsymbol{X}$-measurable random variable and $\tilde{h}_t(z_t)$ is an $\tilde{\boldsymbol{X}}$-measurable random variable. In fact, $\tilde{h}_t(z_t)$ under $\mathbf{Q}$ is probabilistically equivalent to $h_t(z_t)$ under $\mathbf{P}$. It is then clear that the same can be said of

$$H_t(\overline{z}_t) \triangleq \prod_{s=1}^{t} h_s(z_s), \tag{8}$$

and

$$\tilde{H}_t(\overline{z}_t) \triangleq \prod_{s=1}^{t} \tilde{h}_s(z_s).$$

Next observe that $l_t = \tilde{h}_t(\nu_t)$, and that $L_t = \tilde{H}_t(\overline{\nu}_t)$. We are now ready to compute $\mathbf{E_Q}[\, L_t \mid \tilde{\boldsymbol{G}}_t \,]$ in (6) for each $t \geq 1$. (Note that $\mathbf{E_Q}[\, L_0 \mid \tilde{\boldsymbol{G}}_0 \,] = \mathbf{E_Q}[\, L_0 \,] = 1$). Observe that $L_t = \tilde{H}_t(\overline{\nu}_t)$ is an $\tilde{\boldsymbol{X}}$-measurable function of a $\tilde{\boldsymbol{G}}_t$-measurable random vector. Since $\tilde{\boldsymbol{X}}$ and $\tilde{\boldsymbol{G}}_t$ are independent under $\mathbf{Q}$, it follows that

$$\mathbf{E_Q}[\, \tilde{H}_t(\overline{\nu}_t) \mid \tilde{\boldsymbol{G}}_t \,] = \mathbf{E_Q}[\, \tilde{H}_t(\overline{z}_t) \,] \,\big|_{\overline{z}_t = \overline{\nu}_t} .$$

From the remarks preceding equation (8), we have

$$\mathbf{E}_{\mathbf{Q}}\,[\, L_t \mid \tilde{\mathcal{G}}_t \,] = \mathbf{E}\,[\, H_t(\bar{z}_t)\,]\,|_{\,\bar{z}_t = \bar{\nu}_t}.$$

We now set

$$f_t(\bar{z}_t) \triangleq \mathbf{E}\,[\, H_t(\bar{z}_t)\,]. \tag{9}$$

Note that $f_t(\bar{z}_t)$ is a deterministic function of $\bar{z}_t$. Equation (6) becomes

$$\tilde{\mathbf{E}}\,[\, \nu_t(k) \mid \tilde{\mathcal{G}}_{t-1} \,] = \frac{\mathbf{E}_{\mathbf{Q}}\,[\, \nu_t(k)\, f_t(\bar{\nu}_t) \mid \tilde{\mathcal{G}}_{t-1} \,]}{f_{t-1}(\bar{\nu}_{t-1})}.$$

We can write

$$\mathbf{E}_{\mathbf{Q}}\,[\, \nu_t(k)\, f_t(\bar{\nu}_t) \mid \tilde{\mathcal{G}}_{t-1} \,] = \mathbf{E}_{\mathbf{Q}}\,[\, \nu_t(k)\, f_t(\bar{\nu}_{t-1}, \nu_t) \mid \tilde{\mathcal{G}}_{t-1} \,].$$

Now, equation (1) implies that ν_t is independent of $\tilde{\mathcal{G}}_{t-1}$ under $\mathbf{Q}$. Therefore,

$$\mathbf{E}_{\mathbf{Q}}\,[\, \nu_t(k)\, f_t(\bar{\nu}_t) \mid \tilde{\mathcal{G}}_{t-1} \,] = \mathbf{E}_{\mathbf{Q}}\,[\, \nu_t(k)\, f_t(\bar{z}_{t-1}, \nu_t)\,]\,|_{\,\bar{z}_{t-1} = \bar{\nu}_{t-1}}.$$

Since simultaneous events do not occur, it is trivial to compute

$$\begin{aligned}\mathbf{E}_{\mathbf{Q}}\,[\, \nu_t(k)\, f_t(\bar{z}_{t-1}, \nu_t)\,] &= f_t(\bar{z}_{t-1}, e_k)\mathbf{Q}(\, \nu_t(k) = 1\,)\\ &= \mu_t(k)\, f_t(\bar{z}_{t-1}, e_k),\end{aligned}$$

where e_k is the standard unit vector in $\mathbf{R}^K$ with a 1 in the k th position and a 0 in the other $K-1$ positions. We conclude that

$$\tilde{\mathbf{E}}\,[\, \nu_t(k) \mid \tilde{\mathcal{G}}_{t-1} \,] = \frac{\mu_t(k) f_t(\bar{\nu}_{t-1}, e_k)}{f_{t-1}(\bar{\nu}_{t-1})},$$

and hence,

$$\mathbf{E}\,[\, n_t(k) \mid \mathcal{G}_{t-1} \,] = \frac{\mu_t(k) f_t(\bar{n}_{t-1}, e_k)}{f_{t-1}(\bar{n}_{t-1})},$$

with the obvious meaning of the symbol $\bar{n}_{t-1}$.

V. Summary

We have shown that if $\{ n_t(k);\ t \geq 1,\ 1 \leq k \leq K \}$ is a discrete-time, multivariate point process residing in the probabilistic setting outlined in Section II, then

$$\mathbf{E} [\ n_t(k)\ |\ \mathcal{G}_{t-1}\ ] = \frac{\mu_t(k) f_t(\bar{n}_{t-1},\ e_k)}{f_{t-1}(\bar{n}_{t-1})},$$

where $f_t(\bar{z}_t)$ is given by equations (7), (8), and (9).

The $\mu_t(k)$ introduced in Section III were arbitrary; if we set $\mu_t(k) = \dfrac{1}{K+1}$ for all t and all k in the preceding equations, we find that

$$\mathbf{E} [\ n_t(k)\ |\ \mathcal{G}_{t-1}\ ] = \frac{a_t(\bar{n}_{t-1},\ e_k)}{a_{t-1}(\bar{n}_{t-1})},$$

where

$$a_t(\bar{z}_t) \triangleq \mathbf{E} [\ \prod_{s=1}^{t} \left\{ \sum_{k=1}^{K} (\lambda_s(k) - p_s) n_s(k) + p_s \right\}].$$

Appendix

For $1 \leq s \leq t$, let z_s denote either the zero vector in $\mathbb{R}^K$ or any standard unit vector in $\mathbb{R}^K$. In this appendix we prove that if $E \in \tilde{\mathcal{X}}$, then (using the notation of Sections III and IV)

$$\mathbf{Q}(\ \nu_t = z_t,\ \ldots,\ \nu_1 = z_1,\ E\ ) = \left\{ \prod_{s=1}^{t} \mathbf{Q}(\ \nu_s = z_s\ ) \right\} \mathbf{Q}(E).$$

Since $\tilde{\mathcal{G}}_t = \sigma\{ \nu_1,\ \ldots,\ \nu_t \}$, this will prove that $\tilde{\mathcal{G}}_t$ and $\tilde{\mathcal{X}}$ are independent under $\mathbf{Q}$.

Proof. Using the definition of conditional probability,

$$\mathbf{Q}(\ \nu_t = z_t,\ \ldots,\ \nu_1 = z_1,\ E\ ) = \int_{\{ \nu_{t-1} = z_{t-1},\ \ldots,\ \nu_1 = z_1,\ E \}} \mathbf{Q}(\ \nu_t = z_t\ |\ \tilde{\mathcal{F}}_{t-1}\ )\ d\mathbf{Q}. \quad (10)$$

Since simultaneous events do not occur, if $z_t = e_k$, $Q(\nu_t = z_t \mid \bar{F}_{t-1}) = \mu_t(k)$. If $z_t = 0$, $Q(\nu_t = z_t \mid \bar{F}_{t-1}) = 1 - \sum_{k=1}^{K} \mu_t(k)$. Since $Q(\nu_t = z_t \mid \bar{F}_{t-1})$ is deterministic,

$$Q(\nu_t = z_t \mid \bar{F}_{t-1}) = Q(\nu_t = z_t).$$

Hence (10) becomes

$$Q(\nu_t = z_t, \ldots, \nu_1 = z_1, E) = Q(\nu_t = z_t) Q(\nu_{t-1} = z_{t-1}, \ldots, \nu_1 = z_1, E).$$

The remainder of the proof by induction is clear.

QED

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